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(6) EQUIVALENCE OF INFORMATION PATTERNS
AND ESSENTIALLY DETERMINATE GAMES,

(10) by Norman Dalkey,

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M. C.

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EQUIVALENCE OF INFORMATION PATTERNS AND ESSENTIALLY DETERMINATE GAMES

Norman Dalkey

Summary: ~~(1)~~ A necessary and sufficient condition is derived for the equivalence of information patterns in general games. ~~(2)~~ Calling a general game *essentially determinate* if it has an equilibrium point in pure strategies for every possible pay-off function, a necessary and sufficient condition for essential determinateness is derived in terms of the information pattern.

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EQUIVALENCE OF INFORMATION PATTERNS AND ESSENTIALLY DETERMINATE GAMES

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Introduction

In the first sections (1-5) we examine the equivalence of games in extensive form, using the model proposed by Kuhn (1). There are several kinds of equivalence that might be explored, depending in part on what one considers reasonable methods of play. We are concerned with equivalence with respect to mixed strategies. A quite different notion of equivalence would be needed, for example, if the play were limited to behavior strategies.[1]

The notion of equivalence we evolve is only distantly related to the idea of strategic equivalence introduced by von Neumann and Morgenstern (5 p 245-248). They are concerned mainly with variations in the payoff function which leave a solution invariant; we shall be concerned with variations in the structure of a game in extensive form which leave the major strategic properties of the game invariant irrespective of the pay-off function[2].

A complete treatment of equivalence under variations in the structure of a game in extensive form is not given, but only equivalence under variations in the pattern of information. Roughly speaking two information patterns for the same player are equivalent if they differ at a given position in the game only in the knowledge which that player has of his own previous moves.

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1. cf. Kuhn (1)
2. The mode of approach is quite similar to that of Krentel, Quine, McKinsey (2), and our results may be considered as an extension to general games of their results for two-person, zero-sum games.

In the later sections, these results are applied to furnish a necessary and sufficient condition for a general game to have an equilibrium point in pure strategies independently of the particular pay-off function or of the particular probability distributions assigned to chance moves. We call such games essentially determinate, since the question whether or not they will have an equilibrium point in pure strategies is completely determined by the information pattern alone.

The condition, which we have labelled effectively perfect information, is that at any move a player know all preceding moves of his opponents, and know at least as much as his opponents knew when they made those moves. In the special case that there are no chance moves, this condition is simply that the game be equivalent with respect to information to a game of perfect information.[1]

2. Games in Extensive Form

We shall follow rather closely the definition of games in extensive form given by Kuhn, with some minor notational modifications.

Definition 1. A general n-person game Γ in extensive form is defined by

P1. A game tree, K, which is an ordered set of positions $\{x,y,z\dots\}$.

P2. An information pattern $\mathfrak{U} = \{\mathfrak{U}_0, \mathfrak{U}_1, \dots \mathfrak{U}_n\}$ where each $\mathfrak{U}_i = \{U, V, \dots\}$ partitions a subset of K.

P3. An n-component, real vector function $h = (h_1(w), h_2(w), \dots, h_n(w))$ defined on a designated subset W of K.

P4. A set function $p(\nu, U)$, defined on $U \varepsilon \mathfrak{U}_1$, $\nu = 1, 2, \dots, m(U)$, $0 < p(\nu, U) \leq 1$.

When we wish to indicate the dependency of Γ on these entities, we shall write

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[1] von Neumann and Morgenstern first proved that perfect information is a sufficient condition for a two-person zero sum game to have a saddlepoint in pure strategies (see 5, Sec. 15). Kuhn extended this result to equilibrium points in pure strategies for general games. Shapley (4) gave a necessary and sufficient condition for a saddlepoint in pure strategies for a restricted class of two-person zero sum games which is similar to the condition for general games we give below.

$$\Gamma = \Gamma(K, \mathfrak{U}, h, p) \; .$$

Rather than axiomatize this set of primitives, we shall follow Kuhn in giving them a geometrical interpretation.

K is a finite tree, embedded in an oriented Euclidean plane, with a distinguished vertex O . The set W of end points of K are called plays, the remaining vertices moves. (We shall sometimes call both plays and moves by the common name positions.) The unique unicursal path leading from O to a play w will also be called a play, and represented by $\underline{w}$.

The m positions immediately succeeding a position x are indexed by positive integers $\nu = 1, 2, \ldots, m$ where m depends on x. x_ν will designate the ν'th position immediately following x. m(x) designates the total number of possible choices (alternatives) at x. The rank of x in K, i.e., the number of positions which preceed x, will be designated by r(x). D(x) (descendants of x) will represent the set of all positions that follow x, and $D(\nu, x)$ the set which follow by the ν'th alternative.

The information pattern $\mathfrak{U}$ first partitions the moves of K into n+1 exclusive subsets, and then further subdivides each of these subsets into information sets. $\sum_{U \in \mathfrak{U}_i} U = P_i$ are those positions where player i "has the move". $P = \{P_0, P_1, \ldots, P_n\}$ is called the player partition. At each $x \in U \in \mathfrak{U}_i$, player i is informed that he is at one of the positions in U. For each information set U, the number of alternatives is the same for all positions in U, hence we can write m(U), the total number of alternatives available at any position in U. Information sets are further restricted by the rule that no information set intersects the same play more than once.

The set $\mathfrak{U}_0$ is reserved for the "chance player", i.e., every $x \in P_0$ is a chance move. We allow $U \in \mathfrak{U}_0$ to contain more than one move; this furnishes a convenient way of identifying the probability distributions at different chance moves.

$p(\nu,U)$ is assigned so that for $U \notin \underline{\mathfrak{U}}_0$, $p(\nu,U) = 1$ for every ν. For $U \in \underline{\mathfrak{U}}_0$, $0 < p(\nu,U) < 1$, $\sum_{\nu=1}^{m(U)} p(\nu,U) = 1$. Hence for chance moves, $p(\nu,U)$ assigns a probability distribution over the m possibilities. We set $p(\nu,x) = p(\nu,U)$, $x \in U$.

h is the pay-off function. To each play w it assigns an n-tuple of real numbers determining how much each player is to receive at that point.

We shall call a pair $(K,\underline{\mathfrak{U}})$ a <u>game structure</u>, and correspondingly $(K,\underline{\mathfrak{U}})$ will be called <u>the structure</u> of $\Gamma(K,\underline{\mathfrak{U}},h,p)$. $\underline{\mathfrak{U}}_i$ will be called the <u>information pattern for player</u> i.

<u>Definition 2</u>. A <u>pure strategy</u> for player i is a function $\pi_i(U)$ which maps each $U \in \underline{\mathfrak{U}}_i$ onto a positive integer $\nu \leq m(U)$. (We shall sometimes also use ρ_i, τ_i to denote pure strategies). We define the <u>choice at a position</u> x of a strategy π_i as $\pi_i(x) = \pi_i(U)$ where $x \in U$.

It is clear that a game structure $(K,\underline{\mathfrak{U}})$ completely determines the set of all possible pure strategies (as well as mixed and behavior strategies) for each personal player.

Note that our definition specifies strategies for the chance player as well as for the personal players. A chance strategy corresponds to von Neumann and Morgenstern's "umpire's choice." (5 p.81).

Let $\pi^* = (\pi_0,\pi_1,\pi_2,\ldots,\pi_n)$ designate an n+1-tuple of strategies, one for each player, and $\pi = (\pi_1,\pi_2,\ldots,\pi_n)$ an n-tuple of strategies, one for each personal player. It is clear that each π^* determines a unique play w, which we will designate by $w(\pi^*)$. Let $\pi^*(U) = \pi_i(U)$ where $U \in \underline{\mathfrak{U}}_i$ and π_i is contained in π^*. We define $\underline{w}(\pi^*)$ recursively.

<u>Definition 3</u>. $\underline{w}(\pi^*)$ is the set of positions such that

1. $O \in \underline{w}(\pi^*)$
2. $x \in \underline{w}(\pi^*)$ implies $x_\nu \in \underline{w}(\pi^*)$ where $\pi^*(x) = \nu$

$w(\pi^*)$ is the final member of $\underline{w}(\pi^*)$. In case $x \epsilon \underline{w}(\pi^*)$, π^* is said to realize x. More generally, if $\bar{\pi}$ is an m-tuple of strategies $m \leq n+1$, $\bar{\pi}$ is said to realize x if there is some π^* containing $\bar{\pi}$ and $x \epsilon \underline{w}(\pi^*)$.

An immediate consequence of definition 3 is

Lemma 1: $w(\pi^*) = w(\rho^*)$ if and only if for every $x \epsilon \underline{w}(\pi^*)$, $\pi^*(x) = \rho^*(x)$, or, equivalently, if and only if for every U that intersects $\underline{w}(\pi^*)$, $\pi^*(U) = \rho^*(U)$.

Let π^*/ρ_i designate the n-tuple which results when ρ_i is substituted for π_i in π^*; i.e., if $\pi^* = (\pi_0, \pi_1, \ldots, \pi_i, \ldots, \pi_n)$ then $\pi^*/\rho_i = (\pi_0, \pi_1, \ldots, \rho_i, \ldots, \pi_n)$.

Theorem 1: $w(\pi^*/\rho_i) = w(\pi^*/\tau_j)$ implies

$w(\pi^*) = w(\pi^*/\rho_i) = w(\pi^*/\tau_j) = w(\pi^*/\rho_i/\tau_j)$

Proof: If no $U \epsilon \underline{\mathfrak{U}}_i$ intersects $\underline{w}(\pi^*/\rho_i)$, then by definition 3 $w(\pi^*/\rho_i)$ is independent of ρ_i, and hence $w(\pi^*)$ is independent of π_i; thus $w(\pi^*/\rho_i) = w(\pi^*)$. Similarly $w(\pi^*/\tau_j) = w(\pi^*/\tau_j/\rho_i)$. If $U \epsilon \underline{\mathfrak{U}}_i$ intersects $\underline{w}(\pi^*/\rho_i)$, then by hypothesis and lemma 1, $\pi^*/\rho_i(U) = \pi^*/\tau_j(U)$, i.e., $\pi_i(U) = \rho_i(U)$. If no $V \epsilon \underline{\mathfrak{U}}_j$ intersects $\underline{w}(\pi^*/\tau_j)$, then $w(\pi^*/\tau_j)$ is independent of τ_j. If $V \epsilon \underline{\mathfrak{U}}_j$ intersects $\underline{w}(\pi^*/\tau_j)$, then $\pi^*/\rho_i(V) = \pi^*/\tau_j(V)$, i.e., $\pi_j(V) = \tau_j(V)$. Since for $U \epsilon \underline{\mathfrak{U}}_k$, $k \neq i$, $k \neq j$, $\pi^*(U) = \pi^*/\rho_i(U) = \pi^*/\tau_j(U)$, the equalities follow by lemma 1.

Theorem 1 plays no role in the subsequent development. However, it illustrates a form of structural relationship among plays and strategies which is lost by a transformation to the normal form of a game.

In the same way that an n+1-tuple π^* of strategies defines a single play, an n-tuple of strategies π defines a subtree $K(\pi)$, where each branch point is a chance move. The end points of $K(\pi)$ are designated by $W(\pi) = \{w \mid$ there is a π_0 such that $w = w(\pi_0, \pi)\}$

<u>Lemma 2</u>. $W(\pi) = W(\rho)$ if and only if $w(\pi_0,\pi) = w(\pi_0,\rho)$ for every π_0.

<u>Proof</u>: Sufficiency is immediate. Necessity: Assume $W(\pi) = W(\rho)$ but for some π_0, $w(\pi_0,\pi) \neq w(\pi_0,\rho)$. Then there is some Υ_0 such that $w(\pi_0,\pi) = w(\Upsilon_0,\rho)$. But then by lemma 1, for every $U \in \mathcal{U}_0$ that intersects $\underline{w}(\Upsilon_0,\rho)$, $\Upsilon_0(U) = \pi_0(U)$, and for every other U that intersects $\underline{w}(\Upsilon_0,\rho)$, $\pi(U) = \rho(U)$. Whence $\underline{w}(\pi_0,\pi) = \underline{w}(\pi_0,\rho)$, a contradiction.

<u>Definition 4</u>. Let $\underline{w}_x$ designate the subplay extending from 0 to x, then for $x \neq 0, p(x) = \prod_{y \neq \in \underline{w}_x} p(\nu,y)$, $p(0) = 1$.

<u>Definition 5</u>. The <u>expected</u> <u>pay-off</u> $H(\pi)$ for an n-tuple of pure strategies π is defined as

$$H(\pi) = \sum_{w \in W(\pi)} p(w)h(w)$$

In case $\underline{w}$ contains no chance moves, $p(w) = 1$; otherwise $p(w)$ is the product of the probabilities of those alternatives at chance moves in $\underline{w}$ which lead to further members of $\underline{w}$. Note that $p(w)$ depends only on $p(\nu,U), U \in \mathcal{U}_0$ and is independent of the strategies of personal players; however, the relation holds

$$\sum_{w \in W(\pi)} p(w) = 1$$

3. The Reduced Normal Form of a Game

A list of pure strategies for each personal player of Γ and a specification of the expected pay-off $H(\pi)$ for each n-tuple of pure strategies is called by von Neumann and Morgenstern the normal form of Γ. When $H(\pi)$ is expressed in matrix notation as an n-dimensional array it is called the pay-off matrix of Γ.

In the transformation of games from extensive to normal form a certain kind of redundancy often appears, namely duplications in the pay-off matrix of rows or columns ("hyper-rows" in the case of n-person games.) We shall show below that this redundancy is generally the result of superfluous information on the part of one or more players. It is clear that a player loses no strategic advantages if duplications are deleted. This motivates the definition of a reduced normal form of a game. We first make precise the notion of duplication by defining an equivalence relation for strategies.

Definition 6. $\pi_i \equiv \rho_i$ if and only if $H(\pi) = H(\pi/\rho_i)$ for every π containing π_i.

Lemma 3. $\equiv$ is an equivalence relation (transitive, symmetrical and reflexive).

Proof: Immediate from the definition.

Let s_i, called an equivalence strategy, designate an equivalence class of pure strategies for players i, and S_i the set of all such equivalence classes. The cartesian product $S = S_1 \times S_2 \times \ldots \times S_n$ denotes the set of all n-tuples of equivalence strategies. Let $s = (s_1, s_2, \ldots, s_n)$ denote a member of S. Let $H(s) = H(\pi)$ where $\pi \varepsilon s$.

Definition 7. The reduced normal form of a game Γ is a list S_i of equivalence strategies for each personal player i, and a function $H(s)$ assigning an n-tuple of real numbers to each n-tuple of equivalence strategies.

$H(s)$, when expressed as an n-dimensional array, corresponds to a pay-off matrix where repititions of hyper-rows have been deleted.

4. Equivalence of Games

Under the presumption that mixed strategies are to be allowed, all strategical considerations with respect to a game Γ are summed up in the reduced normal form. This leads us to consider two different games as being equivalent if their reduced normal forms are identical except for possible permutations of the lists of strategies.

Definition 8. Γ is equivalent to Γ' -- in symbols, $\Gamma \equiv \Gamma'$-- if and only if there is a one-one correspondence between S_i and S_i' for each $i \neq o$, such that under this correspondence $H(s) = H'(s')$.

Lemma 4. $\equiv$ is an equivalence relation for games.

Proof: Immediate from the definition (since the one-one correspondences and equality of pay-off are equivalence relations).

5. Equivalence of Information Patterns

Definitions 6 and 8 characterize a kind of equivalence which is not very profound in the sense that it depends on the pay-off functions and the particular probability distributions at chance moves. A more revealing analysis is afforded if we deal with equivalences (to be called essential equivalences) that hold irrespective of the pay-off and probabilities at chance moves.

Consider two game structures $(K;\mathfrak{U})$, $(K';\mathfrak{U}')$ where $K = K'$ and $\mathfrak{U}_0 = \mathfrak{U}_0'$. For convenience, we shall say that p is identical with p' when $p(\nu, U) = p'(\nu, U')$ for every $U \epsilon \mathfrak{U}_0$. This identification seems reasonable in light of the fact that for any $U \notin \mathfrak{U}_0$, $p(\nu, U) = 1$ for every ν. With this convention in mind, we define.

Definition 9. Let $(K;\mathfrak{U})$, $(K';\mathfrak{U}')$ be two game structures where $K = K'$, $\mathfrak{U}_0 = \mathfrak{U}_0'$. $(K;\mathfrak{U})$ is said to be essentially equivalent to $(K';\mathfrak{U}')$ -- in symbols, $(K;\mathfrak{U}) \simeq (K';\mathfrak{U}')$ -- if and only if $\Gamma(K;\mathfrak{U},h,p) \equiv \Gamma(K';\mathfrak{U}',h,p)$ for every h and p for which $\Gamma(K;\mathfrak{U},h,p)$, $\Gamma(K';\mathfrak{U}',h,p)$ are games.

Lemma 5. $\simeq$ is an equivalence relation for game structures.

Proof: From definition 9 and lemma 4.

Theorem 2. $(K;\mathfrak{U}) \simeq (K;\mathfrak{U}')$ implies $P_i = P_i'$ for every i.

Proof: The theorem states that the moves assigned to a given player by $\mathfrak{U}$ must be assigned to the corresponding player by $\mathfrak{U}'$. The theorem holds by assumption (definition 9) for P_0, P_0'. Suppose $P_i \neq P_i'$ for some i. Then there is an $x \epsilon P_i$ such that $x \epsilon P'_k, k \neq i$. We may assume that there are at least two alternatives at x (otherwise x is a trivial move and can be eliminated). Let $x \epsilon U \epsilon \mathfrak{U}_i$ and $x \epsilon V \epsilon \mathfrak{U}'_k$. Let h be as follows:

1) $h_j(w) = 0$ for all w and for all $j \neq i$
2) $h_i(w) = 0$ for all w such that $x_1 \notin \underline{w}$ and $x_2 \notin \underline{w}$.
3) $h_i(w) = 1$ for all w such that $x_1 \epsilon \underline{w}$
4) $h_i(w) = -1$ for all w such that $x_2 \epsilon \underline{w}$

With this h it is immediately clear that we can never have $H_i(\pi) > 0$, $H_i(\rho) < 0$ where $\pi_i = \rho_i$. Let π_k' and ρ_k' be strategies for player k in $(K;\mathfrak{U}')$ such that $\pi_k'(V) = 1$ and $\rho_k'(V) = 2$,

and both π_k' and ρ_k' realize x. Let τ be any n-tuple of strategies that realizes x. $H_i(\tau/\pi_k') > 0$, $H_i(\tau/\rho_k') < 0$, hence there is no equivalence strategy in $\Gamma(K,\underline{\mathfrak{U}},h,p)$ corresponding to the s_i' in $\Gamma(K,\underline{\mathfrak{U}}',h,p)$ containing τ_i.

Theorem 2 assures that there is no loss of generality in specializing to the case $P = P'$. For the next step, we specialize even further, and consider the case of a fixed (but arbitrary) information pattern $\underline{\mathfrak{U}} - \underline{\mathfrak{U}}_i$ for all players but one, and examine the effect of varying $\underline{\mathfrak{U}}_i$. In this uncomplicated case it is convenient to overlook the fact that we are dealing with two different game structures, and identify all the components, except the information pattern and strategies for the player i. With this convention it becomes meaningful to write, for example, π/ρ_i', where π is an n-tuple of strategies originally defined for $(K,\underline{\mathfrak{U}})$ and ρ_i' is a strategy for player i in $(K',\underline{\mathfrak{U}}')$. This convention has the virtue that it saves most of the labor of trivial proofs of one-one correspondences.

<u>Definition 10</u>. Let $(K,\underline{\mathfrak{U}})$, $(K,\underline{\mathfrak{U}}')$ be two game structures where $\underline{\mathfrak{U}}-\underline{\mathfrak{U}}_i = \underline{\mathfrak{U}}'-\underline{\mathfrak{U}}_i'$. A pure strategy π_i for player i in $(K,\underline{\mathfrak{U}})$ is said to be <u>essentially equivalent</u> to a pure strategy π_i' for player i in $(K,\underline{\mathfrak{U}}')$ -- in symbols $\pi_i \simeq \pi_i'$ -- when $H(\pi) = H(\pi/\pi_i')$ for every π containing π_i and every h and p for which $\Gamma(K,\underline{\mathfrak{U}},h,p)$, $\Gamma(K,\underline{\mathfrak{U}}',h,p)$ are games.

<u>Lemma 6</u>. Let $(K,\underline{\mathfrak{U}})$ and $(K,\underline{\mathfrak{U}}')$ be as in definition 10. Then $(K,\underline{\mathfrak{U}}) \simeq (K,\underline{\mathfrak{U}}')$ if and only if for every pure strategy π_i for player i in $(K,\underline{\mathfrak{U}})$ there is a strategy π_i' for player i in $(K,\underline{\mathfrak{U}}')$ such that $\pi_i \simeq \pi_i'$ and vice versa.

Proof: Sufficiency. If $\tau_j \equiv \rho_j, j \neq i$, in $\Gamma(K, \underline{\mathfrak{U}}, h, p)$ then $\tau_j \equiv \rho_j$ in $\Gamma(K, \underline{\mathfrak{U}}', h, p)$. For, assume the contrary, then $H(\pi) = H(\pi/\rho_j)$ for every π containing τ_j, but for some π' containing $\tau_j, H(\pi') \neq H(\pi'/\rho_j)$. Let π_i' be the strategy of the i'th player in π'. By hypothesis there is a π_i such that $H(\pi) = H(\pi/\pi_i')$ for every π containing π_i; In particular $H(\pi'/\pi_i) = H(\pi')$. But by hypothesis $H(\pi'/\pi_i) = H(\pi'/\pi_i/\rho_j)$, since π' contains τ_j and $\tau_j \equiv \rho_j$. But also by hypothesis $H(\pi'/\pi_i/\rho_j) = H(\pi'/\pi_i/\rho_j/\pi_i')$ $= H(\pi'/\rho_j)$. Whence $H(\pi') = H(\pi'/\rho_j)$, which is a contradiction. Similarly, $\tau_j \equiv \rho_j$ in $\Gamma(K, \underline{\mathfrak{U}}', h, p)$ implies $\tau_j \equiv \rho_j$ in $\Gamma(K, \underline{\mathfrak{U}}, h, p)$. Whence, the equivalence strategies S_j in $\Gamma(K, \underline{\mathfrak{U}}, h, p)$ for $j \neq i$ correspond one-for-one with the equivalence strategies S_j' in $\Gamma(K, \underline{\mathfrak{U}}' h, p)$. We define a correspondence $s_i \leftrightarrow s_i'$ by the rule: if $\pi_i \varepsilon s_i$ then $\pi_i' \varepsilon s_i'$ where $\pi_i' \simeq \pi_i$. Clearly, if $\rho_i \equiv \pi_i$, then for every $\rho_i' \simeq \rho_i$, $\rho_i' \equiv \pi_i'$, whence the correspondence is one-one. The equality $H(s) = H(s')$ for this set of correspondences follows immediately from the definition of $\simeq$ for strategies.

Necessity. Suppose that there is a π_i in $(K, \underline{\mathfrak{U}})$ such that there is no π_i' in $(K, \underline{\mathfrak{U}}')$ for which $\pi_i \simeq \pi_i'$. Then there is some h and p and some π containing π_i such that $H(\pi) \neq H(\pi/\pi_i')$. This violates the assumption that $(K, \underline{\mathfrak{U}}) \simeq (K, \underline{\mathfrak{U}}')$.

Theorem 3. Let $(K, \underline{\mathfrak{U}})$, $(K, \underline{\mathfrak{U}}')$ be two game structures with $\underline{\mathfrak{U}} - \underline{\mathfrak{U}}_i = \underline{\mathfrak{U}}' - \underline{\mathfrak{U}}_i'$. The following are necessary and sufficient conditions for $\pi_i \simeq \rho_i$, where π_i is a strategy for player i in $(K, \underline{\mathfrak{U}})$ and ρ_i is a strategy for player i in $(K, \underline{\mathfrak{U}}')$.

1) $W(\pi) = W(\pi/\rho_i)$ for every π containing π_i.

2) $w(\pi^*) = w(\pi^*/\rho_i)$ for every π^* containing π_i.

Proof: (i) 1) implies $\pi_i \simeq \rho_i$. This is immediate, since if $W(\pi) = W(\pi/\rho_i)$ then $H(\pi) = H(\pi/\rho_i)$ for every π.

(ii) $\pi_i \simeq \rho_i$ implies 2). Suppose there is some π^* containing π_i such that $w(\pi^*) \neq w(\pi^*/\rho_i)$. Then there is no τ^* such that $w(\tau^*/\rho_i) = w(\pi^*)$, otherwise, by definition 3, $\tau^*/\rho_i(x) = \pi^*(x)$ for every $x \epsilon \underline{w}(\pi^*)$ and hence, by definition 3 again $w(\tau^*/\rho_i) = w(\pi^*/\rho_i) = w(\pi^*)$. Let p be arbitrary and set $h_i(w) = 0$ for every $w \neq w(\pi^*)$, and set $h_i(w(\pi^*)) = 1$. Then $H_i(\pi) > 0$ for some π containing π_i, whereas $H_i(\pi/\rho_i) = 0$.

(iii) 2) implies 1). Immediate.

Definition 11. A set of positions B(not necessarily an information set) is said to be realizable by π_i if there is a π^* containing π_i and B intersects $\underline{w}(\pi^*)$. Let $U \epsilon \mathfrak{U}_i$ be an information set and B a subset of U. B is said to be isolated in U when, for every π_i, if B is realizable by π_i, then U-B is not realizable by π_i.

Lemma 7. Let $U\nu$ designate the set of positions which follow any move in U by choice of the ν'th alternative. B is isolated in $U \epsilon \mathfrak{U}_i$ if and only if for every $x \epsilon B$, $y \epsilon U-B$, there is a $V \epsilon \mathfrak{U}_i$ such that $x \epsilon V\nu, y \epsilon V\eta$ and $\nu \neq \eta$.

Proof: Sufficiency. Consider any π_i, and assume π_i realizes some $x \epsilon B$ and $y \epsilon U-B$. By hypothesis, there is a $V \epsilon \mathfrak{U}_i$ such that $x \epsilon V\nu, y \epsilon V\eta, \nu \neq \eta$. But if π_i realizes x, then $\pi_i(V) = \nu$ by definition 3, whereas if π_i realizes y, then $\pi_i(V) = \eta$, which contradicts the definition of strategy.

Necessity. Assume that for some $x \epsilon B, y \epsilon U-B$, for every $V \epsilon \mathfrak{U}_i$ such that $x \epsilon V\nu, y \epsilon V\eta$ then $\nu = \eta$. Let π_i realize x and ρ_i

realize y. We construct a $\mathcal{T}_i$ so that for every $V \epsilon \underline{\mathcal{U}}_i$ such that $\underline{w}_x$ intersects $V, \mathcal{T}_i(V) = \pi_i(V)$ and for every $V \epsilon \underline{\mathcal{U}}_i$ such that $\underline{w}_y$ intersects $V, \mathcal{T}(V) = \rho_i(V)$. Then $\mathcal{T}_i$ realizes both x and y, hence B is not isolated in U.

<u>Definition 12</u>. a) Let $(K;\underline{\mathcal{U}})$, $(K;\underline{\mathcal{U}}')$ be two game structures, where $P_i = P_i' . \underline{\mathcal{U}}_i$ is said to be an <u>immediate inflation</u> of $\underline{\mathcal{U}}_i'$ when there is a $V \epsilon \underline{\mathcal{U}}_i'$ and $U_1, U_2 \epsilon \underline{\mathcal{U}}_i$ such that $\underline{\mathcal{U}}_i - \{U_1, U_2\} = \underline{\mathcal{U}}_i' - \{V\}$ and U_1, U_2 are isolated in V.

b) $\underline{\mathcal{U}}_i$ is an <u>inflation</u> of $\underline{\mathcal{U}}_i'$ when there is a finite sequence $\underline{V}_i^1, \underline{V}_i^2, \ldots, \underline{V}_i^l$ such that $\underline{\mathcal{U}}_i = \underline{\mathcal{V}}_i^l$ and $\underline{\mathcal{U}}_i' = \underline{\mathcal{V}}_i^1$ and $\underline{\mathcal{V}}_i^{j+1}$ is an immediate inflation of $\underline{\mathcal{V}}_i^j$, $j = 1, 2, \ldots, l - 1$.

c) $\underline{U}_i$ is <u>completely inflated</u> when there is no $\underline{\mathcal{V}}_i$ such that $\underline{\mathcal{V}}_i$ is an inflation of $\underline{\mathcal{U}}_i$.

d) $\underline{\mathcal{U}}_i$ is a <u>complete inflation</u> of $\underline{\mathcal{U}}_i'$ if $\underline{\mathcal{U}}_i$ is an inflation of $\underline{\mathcal{U}}_i'$ and $\underline{\mathcal{U}}_i$ is completely inflated.

e) $\underline{\mathcal{U}}$ is an inflation (complete inflation) of $\underline{\mathcal{U}}'$ when $\underline{\mathcal{U}}_i$ is an inflation (complete inflation) of $\underline{\mathcal{U}}_i'$ for every i.

<u>Theorem 4</u>. Let $(K;\underline{\mathcal{U}})$, $(K;\underline{\mathcal{U}}')$ be two game structures where $\underline{\mathcal{U}} - \underline{\mathcal{U}}_i = \underline{\mathcal{U}}' - \underline{\mathcal{U}}_i'$, and $\underline{\mathcal{U}}_i$ is an immediate inflation of $\underline{\mathcal{U}}_i'$, then $(K;\underline{\mathcal{U}}) \simeq (K;\underline{\mathcal{U}}')$.

<u>Proof</u>: Let $V \epsilon \underline{\mathcal{U}}_i'$, $U_1, U_2 \epsilon \underline{\mathcal{U}}_i$ be the sets required by Definition 12a. Let π_i' be any strategy for player i in $(K;\underline{\mathcal{U}}')$. Let π_i be the strategy in $(K;\underline{\mathcal{U}})$ such that for every $V' \neq V$, $\pi_i(V') = \pi_i'(V')$ and $\pi_i(U_1) = \pi_i(U_2) = \pi_i'(V)$, then $\pi_i \simeq \pi_i'$; for, let π^* contain π_i, then $w(\pi^*) = w(\pi^*/\pi_i')$ by lemma 1, and hence $\pi_i \simeq \pi_i'$ by theorem 3. Consider any π_i in $(K;\underline{\mathcal{U}})$. If π_i does not realize V, then there is a π_i', $\pi_i'(U) = \pi_i(U)$ for $U \neq V$, and by lemma 1

$\pi_i \simeq \pi_i'$. If π_i realizes V, then, since U_1 and U_2 are isolated in V, π_i can realize either U_1 or U_2, but not both. Let it realize U_1 and let $\pi_i'(U) = \pi_i(U)$ for every $U \neq V$, and $\pi_i'(U_1) = \pi_i(V)$. Since, by lemma 1, π_i' does not realize U_2, $\pi_i' \simeq \pi_i$. Whence, the theorem follows by lemma 6.

<u>Corollary 4a</u>. If $(K;\underline{\mathfrak{U}})$, $(K;\underline{\mathfrak{U}}')$ are identical game structures except that $\underline{\mathfrak{U}}_i$ is an inflation of $\underline{\mathfrak{U}}_i'$ for some i, then $(K;\underline{\mathfrak{U}}) \simeq (K;\underline{\mathfrak{U}}')$.

<u>Proof</u>: Theorem 4 and lemma 5.

<u>Corollary 4b</u>. If $(K;\underline{\mathfrak{U}})$, $(K;\underline{\mathfrak{U}}')$ are game structures such that for every i either $\underline{\mathfrak{U}}_i = \underline{\mathfrak{U}}_i'$ or $\underline{\mathfrak{U}}_i$ is an inflation of $\underline{\mathfrak{U}}_i'$ or $\underline{\mathfrak{U}}_i'$ is an inflation of $\underline{\mathfrak{U}}_i$, then $(K;\underline{\mathfrak{U}}) \simeq (K;\underline{\mathfrak{U}}')$.

<u>Proof</u>: By repeated applications of corollary 4a.

Theorem 5: If $(K,\underline{\mathfrak{U}}) \simeq (K,\underline{\mathfrak{U}}')$ where $\underline{\mathfrak{U}}-\underline{\mathfrak{U}}_i = \underline{\mathfrak{U}}'-\underline{\mathfrak{U}}_i'$, and both $\underline{\mathfrak{U}}$ and $\underline{\mathfrak{U}}'$ are completely inflated, then $\underline{\mathfrak{U}}_i = \underline{\mathfrak{U}}_i'$. (Two equivalent completely inflated information patterns are identical.)

Proof: Let $(K,\underline{\mathfrak{U}})$, $(K,\underline{\mathfrak{U}}')$ be as in the hypothesis, and assume $\underline{\mathfrak{U}}_i \neq \underline{\mathfrak{U}}_i'$. There is no loss of generality in assuming that there is a $V \in \underline{\mathfrak{U}}_i'$ and $U_1, U_2 \in \underline{\mathfrak{U}}_i$ such that both U_1 and U_2 intersect V. Consider any strategy π_i' in $(K,\underline{\mathfrak{U}}')$. π_i' cannot realize both $V \cap U_1$ and $V \cap U_2$, for suppose it did; then there is a π_i in $(K,\underline{\mathfrak{U}})$ such that $\pi_i \simeq \pi_i'$, hence π_i realizes both $U_1 \cap V$ and $U_2 \cap V$. We define a ρ_i such that for every $U' \in \underline{\mathfrak{U}}_i$, $U' \neq U_1$ and $U' \neq U_2$, $\rho_i(U') = \pi_i(U')$ and $\rho_i(U_1) \neq \rho_i(U_2)$. Now ρ_i realizes both $U_1 \cap V$ and $U_2 \cap V$, and there is no ρ_i' in $(K,\underline{\mathfrak{U}}')$ such that $\rho_i' \simeq \rho_i$, since there is some π^* containing ρ_i such that $w(\pi^*) \in V_\nu$ and another $\bar{\pi}^*$ containing ρ_i such that $w(\bar{\pi}^*) \in V_\eta$. Since U_1 and U_2 are any $U \in \underline{\mathfrak{U}}_i$ that intersect V, each is isolated in V, which contradicts the assumption that $\underline{\mathfrak{U}}_i'$ is completely inflated.

Corollary 5a: An information pattern $\underline{\mathfrak{U}}_i$ for player i has a unique complete inflation.

Proof: Immediate from Theorems 4 and 5.

Theorem 6: $(K,\underline{\mathfrak{U}}) \simeq (K,\underline{\mathfrak{U}}')$ where $\underline{\mathfrak{U}}-\underline{\mathfrak{U}}_i = \underline{\mathfrak{U}}'-\underline{\mathfrak{U}}_i'$ if and only if the complete inflation of $\underline{\mathfrak{U}}_i$ is identical with the complete inflation of $\underline{\mathfrak{U}}_i'$.

Proof: Corollary 5a assures that the unique complete inflations of $\underline{\mathfrak{U}}_i$ and $\underline{\mathfrak{U}}_i'$ exist. Sufficiency follows from theorem 4, necessity from theorem 5.

<u>Corollary 6a</u>: $(K, \underline{\mathfrak{U}}) \simeq (K, \underline{\mathfrak{U}}')$ (where $\mathfrak{U}_0 = \mathfrak{U}_0'$) if and only if the complete inflation of $\underline{\mathfrak{U}}_j$ is identical with the complete inflation of $\underline{\mathfrak{U}}_j'$ for every j.

<u>Proof</u>: Theorem 6 and corollary 4b.

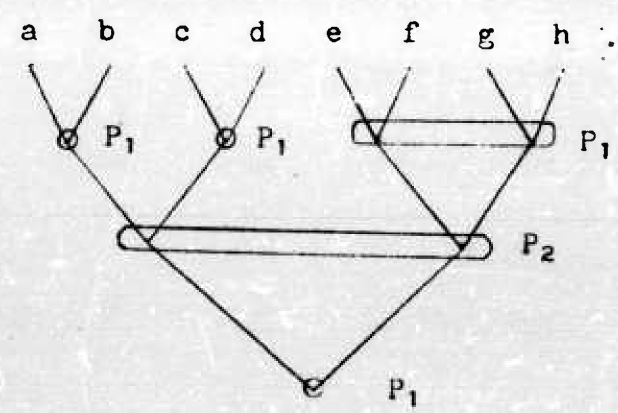
<u>Remark</u>: The rather tedious route whereby we arrive at Theorem 6 and corollary 6a can be bypassed intuitively by the following heuristic discussion. Theorem 4, which is not used directly in the proof, indicates that inflation is essentially a process of adding to a player's information at a given move, some further knowledge about his own previous moves (where this additional information has no implications for previous moves of other players.) But if a player is proceeding according to a preformulated pure strategy, and has complete information about the structure of the game, then at any move where it is his turn to play, his strategy will tell him what previous moves he has made. Hence, information solely about his own previous moves is superfluous. Presumably, a player's memory of a strategy will be equally complete whether it results from a direct choice (pure strategy in the strict sense) or from a prior selection of a pure strategy by some chance device (mixed strategy). However, it is clear that if the player is using behavior strategies — i.e. mixes by some chance selection at each of his moves — there is no guarantee that he will remember all previous selections. This is true particularly where a player is represented by a team of persons, each assigned to different information sets. For this reason — as is readily verifiable — theorem 6 does not hold for behavior strategies.

6. Deflation

So far our emphasis has been on the inflation of information patterns. However, from the practical point of view — i.e., from the point of view of simplifying games -- inflation is not as interesting as its converse, deflation.

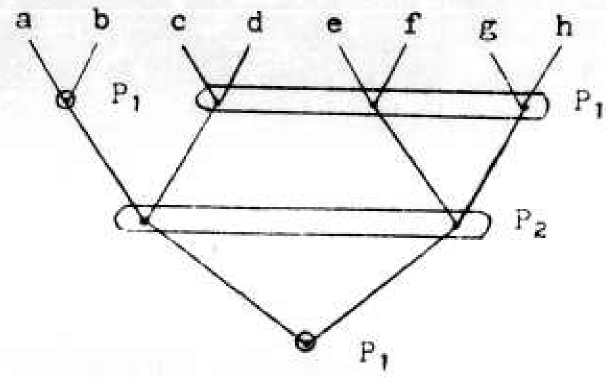
To illustrate the order of reduction in size of pay-off matrices by deflation, in tic-tac-toe the first player has roughly $10^{31,000}$ pure strategies (if we ignore the effect of the stop-rule and assume that each play of the game results in completely filling the open squares.) In essentially reduced form, the first player has only 10^{127} pure strategies. Although this second figure is still ridiculously large, it is clear that a tremendous reduction in size of the game is achieved. The pleasant feature is that this reduction can be carried out by operations directly on the game structure, without first constructing a cosmic sized matrix and then eliminating duplications.

Unfortunately, we cannot assert that every information pattern has a unique complete deflation. Consider the game Γ illustrated below. The ellipses represent information sets for the indicated players.

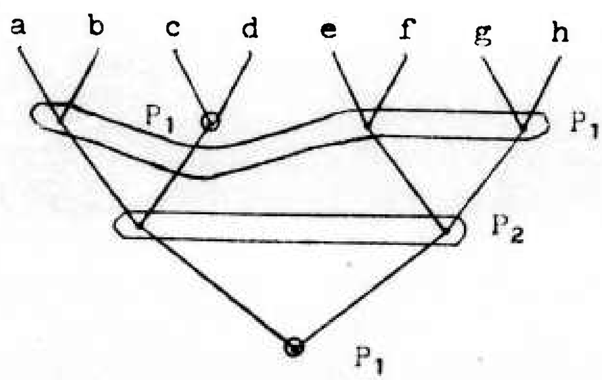


Γ

As it stands $\underline{\mathfrak{U}}_1$ is inflated; player 1 at his second move knows which first move he made. In fact, $\underline{\mathfrak{U}}_1$ is an inflation of $\underline{\mathfrak{U}}_1'$, $\underline{\mathfrak{U}}_1''$ in Γ' and Γ'' respectively.



Γ'



Γ''

Both $\mathfrak{U}_1'$ and $\mathfrak{U}_1''$ are completely deflated. It is easy to verify that in this case there remains a certain amount of redundancy in the information pattern, in the sense that in both Γ' and Γ'' there are two or more strategies that are essentially equivalent.

8. Essentially Determinate Games

In the preceeding sections we were guided by a more or less intuitive notion of equivalence, based on the presumption that the normal form of a game completely exhausts the strategic possibilities. It would be desirable to base our investigations on a set of rules for playing a game in a rational fashion. Unfortunately at present the rules for rational play can be considered as being in a satisfactory state only for one-person and two-person zero-sum games. There is, however, a form of weak solution for general games that has been suggested by Nash (3) which at least can be used for exploring some of the properties of information patterns, and which, in the zero-sum two-person case reduces to the accepted mini-max rule. This is the notion of equilibrium point. Since we have defined only pure strategies above, we limit ourselves to equilibrium points in pure strategies.

Definition 13: A pure strategy n-tuple π is an equilibrium point if and only if $H_i(\pi) \geq H_i(\pi/\rho_i)$ for every i and every ρ_i

Definition 14: $(K,\mathfrak{U})$ is said to be essentially determinate if $\Gamma(K,\mathfrak{U},h,p)$ has an equilibrium point in pure strategies for every h and p for which $\Gamma(K,\mathfrak{U},h,p)$ is a game.

Lemma 8. If $(K,\mathfrak{U}) \simeq (K,\mathfrak{U}')$ then $(K,\mathfrak{U})$ is essentially determinate if and only if $(K,\mathfrak{U}')$ is essentially determinate.

Proof: Immediate from the definitions of $\simeq$ and essentially determinate.

Kuhn has shown (1) that a sufficient condition for a game structure $(K, \mathfrak{U})$ to be essentially determinate is that $(K, \mathfrak{U})$ have <u>perfect information</u>. [1]

<u>Definition 15</u>: $(K, \mathfrak{U})$ is said to have <u>perfect information</u> if every information set U is a unit set.

It is clear that no essential modification results if we restrict the information sets in definition 15 to personal information sets.

It is easy to find examples of games which are essentially determinate and which do not have perfect information. In fact, any one-person game is essentially determinate irrespective of the nature of the information pattern (see lemma 10 below). Hence, perfect information is not a necessary condition for essential determinateness.

Let $U<V$ mean there is a play w which intersects both U and V and the intersection of U and w preceeds the intersection of V and w. We shall also use the notation occasionally $x<U$, which means there is a play w containing x that intersects U, and x preceedes the intersection of U and w.

<u>Definition 16</u>: $(K, \mathfrak{U})$ is said to have <u>effectively perfect information</u> when, for every pair of personal information sets U, V such that $U \in \mathfrak{U}_i$, $V \in \mathfrak{U}_k$, $k \neq i$, if $U<V$ then $V<U_\nu$ for some ν.

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[1] Strictly speaking, he proved something slightly different from this; but the step to showing essential determinateness is trivial, amounting, in fact, only to pointing out that both h and p can be chosen arbitrarily.

Note that the definition says nothing about the relation of information sets of the same player, nor about the relation of personal information sets to chance moves. In intuitive language, a player is said to have effectively perfect information if at any move where it is his "turn" he "remembers" every previous move of each of his personal opponents and knows at least as much as the opponents knew when they made those moves.

Lemma 9: Every one-person game has effectively complete recall.

Proof: Immediate from the definition.

Lemma 10: Every one-person game is essentially determinate.

Proof: Since $H(\pi)(= H_i(\pi_i))$ is a function of only one variable over a finite domain, it admits a simple maximum.

We first show that a game with effectively perfect information can be decomposed into a directed set of subgames (ordered by inclusion), each of which can be examined for equilibrium points independently of all succeeding subgames. The notion of subgame involved is somewhat wider than the direct application of definition 1.

Definition 17: Let B be a set of positions (not necessarily an information set) and $D(B)$ the set of all positions that follow any position in B, i.e. $D(B) = \{y \mid \text{there is an } x \varepsilon B \text{ and } y \varepsilon D(x)\}$. Analogously, $D(\nu, B)$ is the set of positions that follow B via the νth alternative, i.e. $D(\nu, B) = \{y \mid \text{there is an } x \varepsilon B \text{ and } y \varepsilon D(\nu, x)\}$. We will write $x \leq U$ to mean $x < U$ or $x \varepsilon U$. By the greatest lower bound (g.l.b.) of B is meant the x of highest rank that is contained in every $\underline{w}$ that intersects B. $V \leq B$ means $V < B$ or V intersects B.

A. A set of positions B is said to define a <u>subgame</u> Γ_B when:

1) For every personal information set $V<B$, $B \subset V_\nu$ for some ν.

2) For every personal information set $V \geq B$, $V \subset D(B) \cup B$.

B. A <u>pure strategy</u> for a player i in Γ_B is a function, π_i^B, that maps every $V \in \underline{\mathfrak{U}}_i$, $V \geq B$, onto an integer $\nu < m(V)$. We shall also use the expression π_i^B to designate the strategy π_i <u>limited to</u> Γ_B (i.e. π_i^B is the strategy for player i in Γ_B such that $\pi_i^B(V) = \pi_i(V)$, $V \geq B$, $V \in \underline{\mathfrak{U}}_i$.)

C. $W(\pi^B) = \{ \underline{w} \mid \underline{w} \in D(B)$ and for every personal move $y \geq B$, $y \in \underline{w}$ implies $y_\nu \in \underline{w}$ where $\pi^B(y) = \nu \}$

D. Let $p(B) = \sum_{y \in B} p(y)$. For $x \geq B$, $p_B(x) = \frac{p(x)}{p(B)}$.

E. $H(\pi^B) = \sum_{w \in W(\pi^B)} p_B(w) h(w)$

A rough justification for calling Γ_B a subgame is the following (a more precise justification is found in lemmas 11 - 13 below): Definition 17 A2 states that B <u>isolates</u> a certain portion of K, namely $B \cup D(B)$; no personal information set overlaps $B \cup D(B)$ and the rest of K. 17 A1 assures that one and only one set of choices on all personal information sets $V<B$ will realize a position in $B \cup D(B)$. Thus, given a π which realizes B, the selection of a strategy within Γ_B can be made independently of the remaining part of Γ. Note that $\{0\}$ defines a subgame $\Gamma_{\{0\}}$, namely, Γ itself.

The significant elements for a subgame Γ_B, namely strategies and expected pay-off, are defined in a manner quite analogous to those for a complete game, with g.l.b. B replacing O. We cannot, however, simply define a subtree, say K_B, which begins at g.l.b. B and let K_B with its information pattern define a subgame structure, since in the first place, there may be many personal information sets between g.l.b. B and B which overlap K_B and the rest of K, and secondly, K_B may be much more extensive than the part of K which is isolated from B onward.

<u>Lemma 11</u>: If B defines a subgame, then for $x, y \in B$, g.l.b. $\{x,y\}$ is a chance move, or B is a unit set.

<u>Proof</u>: Suppose B is not a unit set and $x, y \in B$. Then g.l.b. $\{x,y\} \notin B$. But g.l.b. $\{x,y\}$ cannot belong to a personal information set by definition 17 A1. Whence g.l.b. $\{x,y\}$ is a chance move.

<u>Lemma 12</u>: If B defines a subgame Γ_B, then for every π that realizes B, $H(\pi) = p(B)\, H(\pi^B) + T(\pi)$ where $T(\pi)$ is independent of π^B.

<u>Proof</u>: If π realizes B, then $W(\pi^B) = W(\pi) \cap D(B)$. Thus

$$H(\pi) = \sum_{w \in W(\pi^B)} p(w)h(w) + T(\pi) \quad \text{where}$$

$$T(\pi) = \sum_{w \in W(\pi) - W(\pi^B)} p(w)h(w)\text{, whence}$$

$$H(\pi) = p(B) \sum_{w \in W(\pi^B)} p_B(w)h(w) + T(\pi)$$

$$= p(B)\, H(\pi^B) + T(\pi)$$

If $w \in W(\pi) - W(\pi^B)$, then $\underline{w}$ does not intersect B, and hence, by definition 17 A2, does not intersect any personal $V \geq B$. Whence, if π and ρ are identical for all personal $V \not\geq B$, $T(\pi) = T(\rho)$.

Lemma 13: π **is an equilibrium point for** Γ **if and only** if π^B is an equilibrium point for every Γ_B which is a subgame of Γ, and for which $K(\pi)$ intersects B.

Proof: Sufficiency is immediate since Γ is a subgame of itself. Necessity. If π^B is not an equilibrium point of Γ_B, there is an i and a ρ_i^B such that $H_i(\pi^B | \rho_i^B) > H_i(\pi^B)$. Let ρ_i be identical with π_i except in Γ_B, where it is identical with ρ_i^B. Then by lemma 12 $H_i(\pi | \rho_i) > H_i(\pi)$, and π is not an equilibrium point in Γ.

Definition 18: If U and V **are personal information** sets, U is said to be connected with V when there is a sequence of personal information sets $U^1, U^2, \ldots, U^l$ such that $U = U^1$, $V = U^l$ and for each i there is a $\underline{w}$ that intersects U^i and U^{i+1}, and $U^i \not\subset U^{i+1}_\nu$, $U^{i+1} \not\subset U^i_\nu$ for any ν.

Lemma 14: Connected is an equivalence relation.

Proof: a) Transitivity. If U is connected with V and V is connected with V', then there is a sequence $U^1, U^2, \ldots, U^l$ connecting U and V and another sequence $V^1, V^2, \ldots, V^h$ connecting V and V'. The sequence $U^1, U^2, \ldots, U^l, V^2, \ldots, V^h$ is a sequence of the required sort connecting U and V'.

b) Symmetry. Immediate from the definition.

c) Reflexivity. Immediate.

<u>Definition 19</u>: Denote the set of equivalence classes of information sets under <u>connected</u> by $\mathcal{C} = \{C_1, C_2 \ldots\}$. Let $\sum C = \{x \mid$ there is a $U \varepsilon C$ and $x \varepsilon U\}$, then $B(C) = \{x \mid x \varepsilon \sum C$ and there is no $y \varepsilon \sum C$, $y < x\}$ i.e., $B(C)$ consists of the minimal points of $\sum C$.

<u>Lemma 15</u>: If $C \varepsilon \mathcal{C}$ then $B(C)$ defines a subgame.

<u>Proof</u>: (1) Let U be any personal information set, $U \geq B(C)$, i.e., there is some $\underline{w}$ that intersects U and $B(C)$. Let $\underline{w}$ intersect $B(C)$ in V. If $U \not\subset V_{\nu}$ for some ν, then $U \varepsilon C$, and hence every $\underline{w}$ that intersects U intersects $B(C)$. If $U \subset V_{\nu}$ for some ν then again every $\underline{w}$ that intersects U intersects $B(C)$.

(2) Let U be any personal information set, $U < B(C)$. $U \not\varepsilon C$, since $B(C)$ is minimal. There is a $\underline{w}$ that intersects U and intersects $B(C)$ in say $x \varepsilon V \varepsilon C$, hence $V \subset U_{\nu}$ for some ν, otherwise $U \varepsilon C$. Let V' be any other information set in C. By definition V and V' are connected by a sequence $V^1, V^2, \ldots, V^{\ell}$, $V = V^1$, $V' = V^{\ell}$. We have shown that $V^1 \subset U_{\nu}$ for some ν. Assume that $V^j \subset U_{\nu}$. There is a $\underline{w}_0$ that intersects V^j and V^{j+1}, and every $\underline{w}$ that intersects V^j intersects U; hence, $\underline{w}_0$ intersects U and V^{j+1}. Now $U \not\subset V^{j+1}$ for any η, since $U < B(C)$; whence if $V^{j+1} \not\subset U_{\eta}$ for some η, $U \varepsilon C$. Thus $V^{j+1} \subset U_{\eta}$ for some η, and $\eta = \nu$ since, by assumption, $\underline{w}_0$ follows the ν'th alternative at U. Therefore there is a ν such that for every $V \varepsilon C$, $V \subset U_{\nu}$. A-fortiori, $B(C) \subset U_{\nu}$.

<u>Definition 20</u>: An equivalence class C is said to <u>cover</u> an equivalence class C' when $B(C) < B(C')$ and there is no C'' such that $B(C) < B(C'') < B(C')$.

Lemma 16: If C covers C', then for every $U \varepsilon C'$, $U > B(C)$, and for every $V \varepsilon C$, if there is a $\underline{w}$ that intersects V and $B(C')$, $B(C') \subset V_{\underline{w}}$ for some $\underline{w}$.

Proof: Lemma 15.

Lemma 16 says in effect that if an equivalence class C preceeds another equivalence class C', then $B(C')$ defines a subgame of $\Gamma_{B(C)}$.

Lemma 17: Two different equivalence classes cannot cover the same equivalence class.

Proof: Suppose C and C' both cover C''. By lemma 16 every $\underline{w}$ that intersects $B(C'')$ intersects both $B(C')$ and $B(C)$. Since $B(C)$ and $B(C')$ are not identical, we must have either $B(C) < B(C')$ or $B(C') < B(C)$. In which case either C or C' does not cover C''.

If the first position O of $(K, \underline{\mathfrak{U}})$ is a personal move, the C containing O defines the entire game Γ. If $O \varepsilon P_0$, it is possible that no C determines the entire game. In this case it is convenient to extend the definition of $\mathcal{C}$ so that $\{O\} \varepsilon \mathcal{C}$. In either case, we designate the C containing O by C_0.

Theorem 6: $\mathcal{C}$ is a tree under the relation covers with a distinguished vertex, C_0.

Proof: Let $\succ$ denote the proper ancestral of covers, i.e., $C \succ C'$ if and only if there is a sequence $C_1, C_2, \ldots, C_n$ such that $C = C_1$, $C' = C_n$ and C_i covers C_{i+1}. $\succ$ is transitive by definition and asymmetrical by the a-cyclicity of the ordering relation on K. $C_0 \succ C$ for every $C \neq C_0$. Finally, if $C_1 \succ C$, $C_2 \succ C$, then either $C_1 \succ C_2$ or $C_2 \succ C_1$ by lemma 17.

<u>Lemma 18</u>: If $(K, \underline{\mathfrak{U}})$ has effectively perfect information and $C \in \mathcal{C}$ then $C \subset \underline{\mathfrak{U}}_i$ for some i.

<u>Proof</u>: Let $V \in C$, $V \in \underline{\mathfrak{U}}_i$, be any information set in C. Consider any $U \in C$. U is connected with V by a sequence $U^1, U^2, \ldots, U^h$. But if $U^i \in \underline{\mathfrak{U}}_i$, then $U^{i+1} \in \underline{\mathfrak{U}}_i$ since, by assumption, if $U^{i+1} \notin \underline{\mathfrak{U}}_i$, then either $U^i \subset U^{i+1}$ or $U^{i+1} \subset U^i$, and $U^{i+1} \notin C$.

Lemma 18 shows that in the tree $\mathcal{C}$, the transition from a set of subgames to a covering subgame involves only one player.

<u>Theorem 7</u>: A necessary and sufficient condition that $(K, \underline{\mathfrak{U}})$ be essentially determinate is that the complete inflation of $(K, \underline{\mathfrak{U}})$ have effectively perfect information.

<u>Proof</u>: Sufficiency. The restriction to the complete inflation of $(K, \underline{\mathfrak{U}})$ is required only for necessity. Hence, assume that $(K, \underline{\mathfrak{U}})$ has effectively perfect information. We show sufficiency by exhibiting, for any h and p, a strategy n-tuple $\bar{\pi}$ which is an equilibrium point. By a <u>minimal</u> $C \in \mathcal{C}$ we mean one for which there is no C' such that $C \in C'$. It is clear that a minimal C defines a one-person subgame. To simplify notation, rather than writing $\Gamma_{B(C)}$ and $\pi^{B(C)}$ we will write Γ_C and π^C.

(1) For each minimal C, $\bar{\pi}^C$ ($= \bar{\pi}_i{}^C$ where $C \subset \underline{\mathfrak{U}}_i$) is so chosen that

$$H_i(\bar{\pi}^C) \geq H_i(\bar{\pi}^C / \rho_i{}^C)$$

(2) For any C, $\bar{\pi}^C(V) = \bar{\pi}^{C'}(V)$ for $V \in C'$, $C \in C'$. $\bar{\pi}^C(U)$, $U \in C$, is chosen so that

$$H_k(\bar{\pi}^C) \geq H_k(\bar{\pi}^C / \rho_i{}^C), \quad C \subset \underline{\mathfrak{U}}_k$$

where $\rho_i^C(V) = \bar{\pi}^{C'}(V)$ for $V \epsilon C'$, $C \leftarrow C'$.

To show that any $\bar{\pi}$ constructed according to the above recursive rule is an equilibrium point, let ρ_i be any strategy for player i, and let $\rho_i^1, \rho_i^2, \ldots$ be a sequence of strategies for player i constructed as follows: (a) $\rho_i^1 = \rho_i$, (b) ρ_i^j is identical with ρ_i^{j+1} except for the C -- which we shall call C_j -- of highest rank which intersects $K(\bar{\pi}/\rho_i^j)$ and for which $\rho_i^j(V) \neq \bar{\pi}_i(V)$ for some $V \epsilon C_j$. Note that $C_j \subset \mathfrak{U}_i$. (In case there are two or more C of equal rank of the specified kind, let C_j be any one of these.) (c) For $V \geq B(C_j)$, $\rho_i^{j+1}(V) = \bar{\pi}_i(V)$. Now by lemma 12 and the definition of $\bar{\pi}$

$$H_i(\bar{\pi}/\rho_i^{j+1}) \geq H_i(\bar{\pi}/\rho_i^j)$$

Since there are only a finite number of C's which intersect $K(\pi/\rho_i)$ and the step from ρ_i^j to ρ_i^{j+1} requires a reduction in rank (or at most a finite number of steps before a reduction in rank), the sequence ρ_i^j must conclude with a ρ_i^ℓ such that $K(\bar{\pi}/\rho_i^\ell) = K(\bar{\pi})$, whence $H_i(\bar{\pi}/\rho_i^\ell) = H_i(\bar{\pi})$ and therefore $H_i(\bar{\pi}) \geq H_i(\bar{\pi}/\rho_i)$.

<u>Necessity</u>: Assume that $(K, \mathfrak{U}')$, the complete inflation of $(K, \mathfrak{U})$ does not have effectively perfect information. Let $x_0 \epsilon \bar{U} \epsilon \mathfrak{U}_i'$ be the personal position of lowest rank for which there is $z \epsilon D(x)$, $z \epsilon V \epsilon \mathfrak{U}_j'$, $i \neq j$, $V \not\subset \bar{U}_\nu$ for any ν (if there are several positions of the same minimal rank with this property, let x_0 be a any one of these); and let $x \epsilon \bar{V} \epsilon \mathfrak{U}_k'$, $K \neq i$, be the position of lowest rank in $D(x_0)$ such that $\bar{V} \not\subset \bar{U}_\nu$ for any ν. Since

$x \in D(\eta, x_0)$ for some η, there is a $y \in \bar{V}$ such that $y \notin D(\eta, x_0)$. For simplicity we relabel the persons and alternatives so that $\bar{U} \subset P_1$, $\bar{V} \subset P_2$, $x \in D(1, x_0)$. We distinguish three principal cases:

Case I. $x_0 = \text{g.l.b.} \{x, y\}$.

Let $\bar{W} = D(x) \cup D(y)$ and let $m = \max [r(x_1), r(y_1)]$. Define $h(w)$ as follows, where z is the highest ranking intersection of $\underline{w}$ with $\underline{w}_x$ or $\underline{w}_y$:

(a) $w \notin \bar{W}$

1. $z \in P_0$: $h_i(w) = 0$ for every i.

2. $z \notin P_0$, $z \le \{x_0\}$: $h_i(w) = \frac{r(z)}{p(z)}$ for every i.

3. $z \notin P_0$, $z > \{x_0\}$: $h_i(w) = \frac{r(z)}{p(z)}$ for every $i \neq 1$, $h_1(w) = \frac{m+1}{p(z)}$ for $z \notin P_1$ otherwise $h_1(w) = \frac{p(z)}{p(z)}$.

(b) $w \in \bar{W}$.

1. $i \neq 1$, $i \neq 2$: $h_i(w) = \frac{m}{p(z)}$.

2. $i = 1$ or $i = 2$:

	$h_1(w)$	$h_2(w)$
$w \in D(x_1)$	$\frac{m}{p(x)}$	$\frac{m+1}{p(x)}$
$w \in D(x_2)$	$\frac{m+1}{p(x)}$	$\frac{m}{p(x)}$
$w \in D(y_1)$	$\frac{m+1}{p(y)}$	$\frac{m}{p(y)}$
$w \in D(y_2)$	$\frac{m}{p(y)}$	$\frac{m+1}{p(y)}$

With h as defined, no π is an equilibrium point. There are two possibilities:

A. $K(\pi) \cap \bar{W} = \theta$.

Let $\bar{z} \in U \in \underline{U}'_k$ be the highest ranking intersection of $K(\pi)$ with $\underline{w}_x$ or $\underline{w}_y$; and let ρ_k be identical with π_k except that $\rho_k(U) = y$ where $\bar{z}_y \in \underline{w}_x$ or $\underline{w}_y$. Then $H_k(\pi/\rho_k) \geq r(\bar{z}_y) > r(\bar{z}) = H_k(\pi)$

B. $K(\pi) \cap \bar{W} \neq \theta$.

The pay-off to any player is independent of choices made above $\bar{V}$; hence there are essentially four cases, depending on $\pi_1(\bar{U})$ and $\pi_2(\bar{V})$; these are summed up in the matrices

$H_1(\pi)$

$\pi_1(\bar{U})$ \ $\pi_2(\bar{V})$	1	2
1	m	m+1
2	m+1	m

$H_2(\pi)$

$\pi_1(\bar{U})$ \ $\pi_2(\bar{V})$	1	2
1	m+1	m
2	m	m+1

For any pair of choices $\pi_1(\bar{U})$, $\pi_2(\bar{V})$, there is a choice by one of the players that increases his pay-off.

<u>Case II</u>. $x_0 \neq$ g.l.b. $\{x,y\}$, $\overline{U}$ intersects $\underline{w}_y$.

g.l.b. $\{x,y\} \epsilon P_0$, otherwise x_0 would not be the minimal position for which effectively perfect information later fails. Let $y_0 \epsilon \underline{U}$ be the lowest ranking position in $\underline{w}_y$ such that $\underline{U}$ intersects $\underline{w}_x$ and $\overline{V} \not\subset \underline{U}$ for any ν. Because of the minimality condition on $\overline{V}$, $\underline{U} \epsilon \underline{\mathfrak{U}}_i'$. Let $\overline{W}$, z, and m be as in Case I, and set $t = r(x_0) - r(y_0)$. Define h(w) as in Case I for (a) 1,2 and (b) 1.

(a)3. $z \notin P_0$, $z > \{x_0\}$: $h_i(w) = \frac{r(z)}{p(z)}$ for every $i \neq 1$,

$$z \epsilon \dot{\underline{w}}_x : h_1(w) = \frac{2m+1+t}{p(z)}$$

$$z \epsilon \underline{w}_y : h_1(w) = \frac{2m+1-t}{p(z)} \qquad \text{for } z \notin P_1$$

$$h_1(w) = \frac{r(z)}{p(z)} \text{ for } z \epsilon P_1$$

(b)2.

	$h_1(w)$	$h_2(w)$
$w \epsilon D(x_1)$	$\frac{2m}{p(x)}$	$\frac{2m+1}{p(x)}$
$w \epsilon D(x_2)$	$\frac{2m+1-t}{p(x)}$	$\frac{2m}{p(x)}$
$w \epsilon D(y_1)$	$\frac{2m+1-t}{p(y)}$	$\frac{2m}{p(y)}$
$w \epsilon D(y_2)$	$\frac{2m}{p(y)}$	$\frac{2m+1}{p(y)}$

The proof that no π is an equilibrium point proceeds as in Case I except that a more complicated set of subcases must be examined.

A. $K(\pi) \cap \overline{W} = \Theta$. Let $\overline{z}$ be as in Case IA.

1. $\overline{z} <$ g.l.b. $\{x, y\}$: Argument as in Case IA.

2. $\overline{z} >$ g.l.b. $\{x, y\}$: Let $z^x \varepsilon U^x$ and $z^y \varepsilon U^y$ be the highest ranking intersections of $K(\pi)$ with $\underline{w}_x$ and $\underline{w}_y$ respectively. There are two sub-sub-cases.

(i) Either $U^x \nless z^y$ or $U^y \nless z^x$: Assume $U^x \nless z^y$. Let $U^x \varepsilon \underline{\mathcal{U}}_1'$ and let ρ_i be identical with π_i except that $\rho_i(U^x) = \nu$ where $z^x_\nu \varepsilon \underline{w}_x$.
Then

$$H_i(\pi/\rho_i) \geq r(z^x_\nu) + r(z^y) > r(z^x) + r(z^y) = H_i(\pi)$$

a similar argument holds in case $U^y \nless z^x$.

(ii) $U^x < z^y$ and $U^y < z^x$. In this case both U^x, $U^y \varepsilon \mathcal{U}_1'$, otherwise the minimality conditions on $\overline{U}$ and $\overline{V}$ are violated. Let ρ_1 be identical with π_1 except that for every $U \varepsilon \underline{\mathcal{U}}_1'$, $x > U \geq z^x$, $\rho_i(U) = \nu$ where $x \varepsilon D(\nu, U)$. Then

$$H_1(\pi/\rho_1) \geq 2m + r(y_0) > r(z^x) + r(z^y) = H_1(\pi)$$

The central inequality holds since $r(z^y) - r(y_0) < m$ and $r(z^x) < m$.

B. $K(\pi) \cap \overline{W} \neq \Theta$

At most one of x and y is in $K(\pi)$. Let α denote the one that is in $K(\pi)$ and β denote the other. Then α_0

denotes x_0 or y_0 respectively, and $z^{\beta} \varepsilon U^{\beta} \varepsilon \underline{\mathcal{U}}_i'$ denotes the highest ranking intersection of $K(\pi)$ with $\underline{w}_{\beta}$. Note that $z^{\beta} \leq \{\beta_0\}$. Let w_0 represent any $w \varepsilon K(\pi) \cap \overline{W}$.

1. $z^{\beta} < \{\beta_0\}$. U^{β} does not intersect w_{α} because of the minimality condition on α_0. Let ρ_i be identical with π_i except that $\rho_i(U^{\beta}) = \nu$ where $z_{\nu}^{\beta} \varepsilon \underline{w}_{\beta}$. Then
$$H_i(\pi/\rho_i) \geq p(\alpha)h_i(w_0) + r(z_{\nu}^{\beta}) > p(\alpha)h_i(w_0) + r(z^{\beta}) = H_i(\pi)$$
2. $z^{\beta} = \beta_0$.

   There are four possibilities, depending on the choices at $\overline{U}$, $\underline{U}$, and $\overline{V}$. The apparent eight resulting from two choices at each of three information sets is reduced by the fact that either $\overline{U} = \underline{U}$ or else four of the possibilities give $K(\pi) \cap \overline{W} = \theta$ and are treated above. Label the choices at $\underline{U}$ so that $x \varepsilon D(1,\underline{U})$, $y \varepsilon D(2,\underline{U})$. $\overline{U}$, $\underline{U} \varepsilon \underline{\mathcal{U}}_1'$ because of the minimality condition on $\overline{V}$, therefore the choices at $\underline{U}$, $\overline{V}$ are completely under the control of player 1 and we have the sub-matrices

$H_1(\pi)$

$\pi(\overline{U})$	$\pi(\underline{U})$ \ $\pi(\overline{V})$	1	2
1	1	$2m+r(y_0)$	$2m+1+r(x_0)$
2	2	$2m+1+r(y_0)$	$2m+r(x_0)$

$H_2(\pi)$

$\pi(\bar{U})$	$\pi(\underline{U})$ \ $\pi(\bar{V})$	1	2
1	1	$2m+1+r(y_0)$	$2m+r(y_0)$
2	2	$2m+r(x_0)$	$2m+1+r(x_0)$

For any one of the four possibilities, one of the players can choose a strategy which will increase his pay-off.

Note that no essential modification is needed in the preceeding proof if we relax the minimality condition on x_0, requiring merely that it is the lowest ranking position for which effectively perfect information later fails and belongs to an information set which intersects both $\underline{w}_x$ and $\underline{w}_y$.

Case III. $x_0 \neq$ g.l.b. $\{x,y\}$, $\bar{U}$ does not intersect $\underline{w}_y$.

There is no loss of generality in assuming that for every $U \notin \underline{\mathfrak{U}}_0'$, if U intersects $\underline{w}_y$ and $\underline{w}_x$, then $\{x,y\} \subset U_\nu$ for some ν. This assumption is justified by the following:

1. There is a $z \in \bar{V} - D(\bar{U})$, such that for every $U \in \underline{\mathfrak{U}}_2'$, $U \neq \bar{V}$, that intersects $\underline{w}_x$ and $\underline{w}_y$, $\{x,z\} \subset U_\nu$ for some ν. For assume that for every $z' \in \bar{V}$, $z' \notin D(\bar{U})$, there is a $U \in \underline{\mathfrak{U}}_2'$, $\{x,z\} \not\subset U_\nu$ for any ν. By the minimality

condition on $x, U < x_0$. Whence, by the minimality condition on x_0, $\overline{U} \subset U_\nu$ for some ν, and hence $D(U) \subset U_\nu$ for some ν. But then, for every $z'' \varepsilon D(\overline{U}) \cap \overline{V}$, and every $z' \varepsilon \overline{V} - D(\overline{U})$, there is a $U \varepsilon \mathfrak{U}_2'$ such that $z'' \varepsilon U_\nu$, $z' \varepsilon U_\eta$, $\nu \neq \eta$. Thus by Lemma 7 $D(\overline{U}) \cap \overline{V}$ is isolated in $\overline{V}$, contrary to the hypothesis that $\mathfrak{U}'$ is completely inflated. We can take the z thus proved to exist to be y.

2. There is no $U \notin \mathfrak{U}_0' \cup \mathfrak{U}_1' \cup \mathfrak{U}_2'$ such that U intersects $\underline{w}_x$ and $\underline{w}_y$ and $\{x,y\} \not\subset U_\nu$ for any ν by the minimality conditions on x_0 and x.
3. There is no $U \varepsilon \mathfrak{U}_1'$, $U < x_0$ such that U intersects $\underline{w}_x$ and $\underline{w}_y$ and $\{x,y\} \not\subset U_\nu$ for any ν, by the minimality condition on x_0.
4. If there is a $U \varepsilon \mathfrak{U}_1'$, $U > x_0$ and U intersects $\underline{w}_x$, $\underline{w}_y$, $\{x,y\} \not\subset U_\nu$, then we have essentially Case II as noted at the end of the proof for that section.

Let m and z be as in the previous cases. Let $\underline{W} = \overline{W} \cup D(2, x_0)$.

Define an h as follows:

(a) $w \notin \underline{W}$

1. $z \varepsilon P_0$: $h_i(w) = 0$ for every i
2. $z \notin P_0$, $z \ngtr \{x_0\}$: $h_i(w) = \dfrac{r(z)}{p(z)}$
3. $z \notin P_0$, $z > \{x_0\}$: $h_i(w) = \dfrac{r(z)}{p(z)}$ for $i \neq 1$.

$$h_1(w) = \frac{m+2}{p(z)} \quad \text{for } z \notin P_1$$

$$h_1(w) = \frac{r(z)}{p(z)} \quad \text{for } z \varepsilon P_1$$

(b) $w \in \underline{W}$

1. $h_i(w) = \dfrac{m}{p(z)}$ for every $i \neq 1, i \neq 2$

2.

	$h_1(w)$	$h_2(w)$
$w \in D(2,x_0)$	$\dfrac{m+1}{p(x_0)}$	$\dfrac{m}{p(x_0)}$
$w \in D(x_1)$	$\dfrac{m}{p(x)}$	$\dfrac{m+2}{p(x)}$
$w \in D(x_2)$	$\dfrac{m+2}{p(x)}$	$\dfrac{m}{p(x)}$
$w \in D(y_1)$	$\dfrac{m}{p(y)}$	$\dfrac{m}{p(y)}$
$w \in D(y_2)$	$\dfrac{m}{p(y)}$	$\dfrac{m+1}{p(y)}$

Let $W^* = [D(2,x_0) \cup D(x)] \cap D(y)$. We consider two cases.

A. $K(\pi) \cap W^* = \Theta$

The argument is the same as in Case II A 1, 2(i).

B. $K(\pi) \cap W^* \neq \Theta$

Here there are four cases, depending on the choice at $\bar{U}$ and $\bar{V}$, summed up in the matrices

$H_1(\pi)$

$\pi_1(\bar{U})$ \ $\pi_2(\bar{V})$	1	2
1	2m	2m+2
2	2m+1	2m+1

$H_2(\pi)$

$\pi_1(\bar{U})$ \ $\pi_2(\bar{V})$	1	2
1	2m+2	2m+1
2	2m	2m+1

and for any pair of choices, there is another choice open to one of the players which increases his payoff.

mhb

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